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Ring isomorphisms of Murray–von Neumann algebras

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Outline

1. Intoduction
   - Short history
2. Ring isomorphisms of Murray–von Neumann algebras
   - General form of ring isomorphisms

Isomorphisms of regular rings

In 1930's, motivated by the geometry of lattice of the projections of type II_1 factors, von Neumann built the theory on the correspondence between complemented orthomodular lattices and regular rings. Let us recall one of his achievements [1, Part II, Theorem 4.2], applied to the case of $*$-regular rings. Let $\mathfrak{R}, \mathfrak{R}'$ be $*$-regular rings such that their lattices of projections $L_{\mathfrak{R}}$ and $L_{\mathfrak{R}'}$ are lattice-isomorphic. If $\mathfrak{R}$ has order $n \geq 3$ (which means that it contains a ring of matrices of order n), then there exists a ring isomorphism of $\mathfrak{R}$ and $\mathfrak{R}'$ which generates given lattice isomorphism between $L_{\mathfrak{R}}$ and $L_{\mathfrak{R}'}$.

J. von Neumann, Continuous geometry. Foreword by Israel Halperin, Princeton Mathematical Series, No. 25 Princeton University Press, Princeton, N.J. (1960).

Operator algebras version of von Neumann Theorem

- Let $\mathcal{M}$ be a von Neumann algebra with the lattice of projections $P(\mathcal{M})$. Denote by $S(\mathcal{M})$ the $*$-algebra of all measurable operators affiliated with $\mathcal{M}$. [a]

[a] definitions we shall give later

Theorem 1.1

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras of type II_1 and let $\Phi : P(\mathcal{M}) \to P(\mathcal{N})$ be a lattice isomorphism. Then there exists a unique ring isomorphism $\Psi : S(\mathcal{M}) \to S(\mathcal{N})$ such that $\Phi(l(x)) = l(\Psi(x))$ for any $x \in S(\mathcal{M})$, in particular, $\Phi(p) = l(\Psi(p))$ for any $p \in P(\mathcal{M})$.

Here, $l(x)$ is the left projection of x.

Various isomorphisms of $*$-algebras

Various isomorphisms of $*$-algebras

For $*$-algebras $\mathcal{A}$ and $\mathcal{B}$, a (not necessarily linear) bijection $\Phi : \mathcal{A} \to \mathcal{B}$ is called

- a ring isomorphism if it is additive and multiplicative;
- a real algebra isomorphism if it is a real-linear ring isomorphism;
- an algebra isomorphism if it is a complex-linear ring isomorphism;
- a real $*$-isomorphism if it is a real algebra isomorphism and satisfies $\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathcal{A}$;
- an $*$-isomorphism if it is a complex-linear real $*$-isomorphism.

von Neumann algebras

- Let H be a Hilbert space, $B(H)$ be the $*$-algebra of all bounded linear operators on H, $\mathcal{M}$ be a von Neumann algebra in $B(H)$;
- $P(\mathcal{M})$ the set of all projections in $\mathcal{M}$;
- $e, f \in P(\mathcal{M})$ are called equivalent if there exists an element $u \in \mathcal{M}$ such that $u^*u = e$ and $uu^* = f$;
- $e, f \in \mathcal{M}$ notation $e \precsim f$ means that there exists a projection $q \in \mathcal{M}$ such that $e \sim q \leq f$;
- $p \in \mathcal{M}$ is said to be finite, if it is not equivalent to its proper sub-projection;
- $e \in P(\mathcal{M})$ is abelian, if $e\mathcal{M}e$ is an abelian algebra;
- a finite von Neumann algebra $\mathcal{M}$ without nonzero abelian projections is called of type II_1.

Murray-von Neumann algebras

- Let $\mathcal{M}$ be a von Neumann algebra and let $P(\mathcal{M})$ be a set of all projections in $\mathcal{M}$;
- A linear operator x affiliated with $\mathcal{M}$ is called measurable with respect to $\mathcal{M}$ if $e_{(\lambda,\infty)}(|x|)$ is a finite projection for some $\lambda > 0$.
- A linear operator x affiliated with $\mathcal{M}$ is called locally measurable with respect to $\mathcal{M}$ if there exists a sequence of $\{z_n\}$ of central projections increasing to $\mathbf{1}$ such that $z_n x \in S(M)$.
- Let $S(\mathcal{M})$ (resp.$LS(M)$) be the set of all measurable (resp. locally measurable) operators w.r.t. $\mathcal{M}$;

Murray-von Neumann algebras

- The sets $S(\mathcal{M})$ and $LS(M)$ equipped with the algebraic operations of the strong addition and multiplication and taking the adjoint of an operator, become *-algebras;
- If $\mathcal{M}$ is a finite von Neumann algebra, all linear operators affiliated with $\mathcal{M}$ are automatically measurable, and the algebra $S(\mathcal{M}) = LS(M)$ is referred to as the Murray-von Neumann algebra associated with $\mathcal{M}$.
- In this case an algebra $S(\mathcal{M})$ is regular (in the sence of von Neumann), that is, for every $a \in S(\mathcal{M})$ there exists an element $x \in S(\mathcal{M})$ such that $axa = a$.

Measure topology

- Let τ be a faithful normal finite trace on M. A measure topology t_τ on $S(M)$:

$$N(\varepsilon,\delta) = \left\{x \in S(M) : \tau\left(e_{(\varepsilon,\infty)}(|x|)\right) \leq \delta\right\},$$

  where $\varepsilon, \delta > 0$.
- $(S(M), t_\tau)$ is a complete metrizable topological $*$-algebra.[a]

[a] E. Nelson, J. Funct. Anal. 15 (1974) 103–116.

Example 1.2

- if $M = \ell_\infty$, then $S(M) \cong s \equiv \mathbb{C}^{\aleph_0}$;
- if $M = L_\infty(0,1)$, then $S(M) \cong S(0,1)$;
- if $M = B(H)$, then $S(M) \cong B(H)$.

Question of M. Mori and his conjecture

Mori generalized the above Theorem 1.1 for arbitrary von Neumann algebras which do not admit type I_1 nor I_2 direct summands and asked the following question.

Question 1.3

Let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras. What is the general form of ring isomorphisms from $LS(\mathcal{M})$ onto $LS(\mathcal{N})$?[a]

[a] M. Mori, Lattice isomorphisms between projection lattices of von Neumann algebras, Forum Math. Sigma 8 (2020), Paper No. e49, 19 pp.

- Mori himself gave an answer to the above Question in the case of von Neumann algebras of type I_∞ and III.

Result of M. Mori

Theorem 1.4

Let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras of type I_∞ or III. If $\Phi : LS(\mathcal{M}) \to LS(\mathcal{N})$ is a ring isomorphism, then there exist an invertible element $a \in LS(\mathcal{N})$ and a real $*$-isomorphism $\Psi : \mathcal{M} \to \mathcal{N}$ (which extends to a real $*$-isomorphism from $LS(\mathcal{M})$ onto $LS(\mathcal{N})$) such that $\Phi(x) = a\Psi(x)a^{-1}$ for all $x \in LS(\mathcal{M})$.

In this case Φ and Ψ are called similar.

- Mori conjectured that the representation of ring isomorphisms, mentioned above for type I_∞ and III cases holds also for type II von Neumann algebras.[a]

[a] M. Mori, Lattice isomorphisms between projection lattices of von Neumann algebras, Forum Math. Sigma 8 (2020), Paper No. e49, 19 pp.

A.G.Kusrev's result

- A.G. Kusraev establishes necessary and sufficient conditions for existence of band-preserving non trivial automorphisms in an extended complex f-algebra.
- Let $S(0,1)$ be the algebra of all (classes of) measurable complex-valued functions on $(0,1)$.
- The algebra $S(0,1)$ admits discontinuous in the measure topology algebra automorphisms which identically act on the Boolean algebra $\nabla(S(0,1))$.[a][b][c]

[a] A. G. Kusraev, Automorphisms and derivations in the algebra of complex measurable functions, Vladikavkaz Math. J., 7:3 (2005), 45-49

[b] A. G. Kusraev, Automorphisms and derivations in an extended complex f-algebra, Sib. Math. J. 47 (2006) 97–107.

[c] A. E. Gutman, A. G. Kusraev and S. S. Kutateladze, The Wickstead problem. Sib. Elektron. Mat. Izv. 5 (2008), 293–333.

Type I_n case

- If $\mathcal{M}$ is a von Neumann algebra of type I_n, $n > 1$ with the center $Z(\mathcal{M})$ then $S(\mathcal{M})$ is $*$-isomorphic to the algebra $M_n(Z(S(\mathcal{M})))$, where $Z(S(\mathcal{M})) = S(Z(\mathcal{M}))$;
- in this case each algebra automorphisms Φ of $S(\mathcal{M})$ can be uniquely represented in the form

$$\Phi(x) = a\overline{\Psi}(x)a^{-1}, x \in S(\mathcal{M}),$$

where $a \in S(\mathcal{M})$ is an invertible element and $\overline{\Psi}$ is an extension of an automorphism Ψ of the center $S(Z(\mathcal{M}))$.[a] Mori generalized this result to the case of ring automorphisms and showed that Theorem 1.3 is not true for Type I_n case.

[a] S. Albeverio, S. Ayupov, K. Kudaybergenov, R. Djumamuratov, Automorphisms of central extensions of type I von Neumann algebras, Studia Math. 207 (2011), 1-17.

Center-valued norm

- Let $\mathcal{M}$ be a type I or III von Neumann algebra with the center $Z(\mathcal{M})$.
- For any $x \in LS(\mathcal{M})$ there exist a partition $\{z_i\}_{i\in I}$ of unit in $P(Z(\mathcal{M}))$ and a system of elements $\{x_i\}_{i\in I}$ in $\mathcal{M}$ such that

$$z_i x = z_i x_i$$

for all $i \in I$.
- Hence, for any $x \in LS(M)$ setting

$$\|x\|_{LS(\mathcal{M})} = \inf\{c \in Z(LS(M)) : |x| \leq c\},$$

we obtain a center-valued norm on $LS(\mathcal{M})$. Then $\left(LS(\mathcal{M}), \|\cdot\|_{LS(\mathcal{M})}\right)$ becomes a Banach–Kantorovich space over $Z(LS(\mathcal{M})) \cong S(Z(\mathcal{M}))$.

Outline

General form of ring isomorphisms

The following main result confirms the Conjecture 5.1[1] and answers the above Question 1.3 for the type II_1 case.

Theorem 2.1

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras of type II_1. Then every ring isomorphism from $S(\mathcal{M})$ onto $S(\mathcal{N})$ is similar to a real $*$-isomorphism.[a]

[a] Sh.A. Ayupov, K.K.Kudaybergenov, Ring isomorphisms of Murray-von Neumann algebras, J. Funct. Anal. 280 (2021), no. 5, 108891.

So, only the case of algebras of type II_∞ remained open.

[1] M. Mori, Lattice isomorphisms between projection lattices of von Neumann algebras, Forum Math. Sigma 8 (2020), Paper No. e49, 19 pp.

Jordan $*$-isomorphness of type II_1 von Neumann algebras

Corollary 2.2

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras of type II_1. The projection lattices $P(\mathcal{M})$ and $P(\mathcal{N})$ are lattice isomorphic, if and only if the von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ are real $*$-isomorphic (or equivalently, $\mathcal{M}$ and $\mathcal{N}$ are Jordan $*$-isomorphic).

General form of ring isomorphisms

Theorem 2.3

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras without type $\mathrm{I}_{\mathrm{fin}}$ direct summands. Then every ring isomorphism from $LS(\mathcal{M})$ onto $LS(\mathcal{N})$ is similar to a real $*$-isomorphism.[a]

[a] M. Mori, Ring isomorphisms of type II_∞ locally measurable operator algebras, Bulletin of the London Mathematical Society, 2023, DOI: 10.1112/blms.12880

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